数理解析研究所講究録 1394 巻 2004 年 35-46

Completely Regular Codes in Johnson graph

北海道大学・理学研究科　吉田瞳 (Hitomi Yoshida)
Graduate school of Mathematics,
Hokkaido University

1 Introduction

In this article, we study completely regular codes in some distance regular graphs. Completely regular codes were first studied by Biggs [2] and P. Delsarte [4], but there are not yet many articles on this subject. Recently, research of completely regular codes has been developing with research of Terwilliger algebra.

We consider completely regular codes in a distance regular graph. Martin [7] conjectured that for completely regular codes in a distance regular graph, $\gamma_i \leq \gamma_{i+1}$, $\beta_i \geq \beta_{i+1}$ hold. On the other hand, Koolen [6] showed $\gamma_i < \gamma_{i+1}$, $\beta_i > \beta_{i+1}$ in $H(D,2)$. Furthermore, I conjectured $\gamma_i < \gamma_{i+1}$, $\beta_i > \beta_{i+1}$ in $J(n,d)$. In order to study this conjecture, in this paper we first classify the completely regular codes in Johnson graph $J(4,2)$, $J(5,2)$, $J(6,2)$, $J(6,3)$.

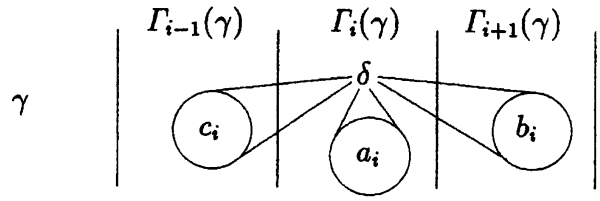
2 Distance regular graphs

A graph is a pair $\Gamma = (V, E)$ consisting of a set V, referred to as the *vertex* set of Γ, and a set E of 2-subsets of V, referred to as the *edge* set of Γ. That is, our graphs are undirected, without loops or multiple edges. We write $\gamma \in \Gamma$ if γ is a vertex of Γ and $\gamma \sim \delta$ if $\{\gamma, \delta\}$ is an edge of Γ. The *distance* $d(\gamma,\delta)$ of two vertices $\gamma, \delta \in \Gamma$ is the length of a shortest path between γ and δ. d $=\max\{d(\gamma,\delta) \mid \gamma,\delta \in \Gamma\}$ is called the *diameter* of Γ. Given $\delta \in \Gamma$, we write $\Gamma_i(\delta)$ for the set of vertices γ with $d(\gamma,\delta) = i$. In particular, $\Gamma(\delta) = \Gamma_1(\delta)$ denotes the set of neighbours of δ. The *valency* $k(\gamma)$ of a vertex γ is the cardinality of $\Gamma(\gamma)$. A graph is called *regular* if each vertex has the same valency k.

Definition 2.1 *A connected graph $\Gamma = (V, E)$ is said to be a distance regular graph* (DRG) *if the numbers*

$$\begin{aligned} c_i &= \mid \Gamma_{i-1}(\gamma) \cap \Gamma(\delta) \mid, \\ a_i &= \mid \Gamma_i(\gamma) \cap \Gamma(\delta) \mid, \textit{and} \\ b_i &= \mid \Gamma_{i+1}(\gamma) \cap \Gamma(\delta) \mid \end{aligned}$$

are independent of the choices of $\gamma, \delta \in \Gamma$ with $d(\gamma, \delta) = i$.



The numbers c_i, a_i and b_i are said to be the *intersection numbers* of Γ.

$$\iota(\Gamma) = \left\{ \begin{array}{cccccc} * & c_1 & \dots & c_{d-1} & c_d \\ a_0 & a_1 & \dots & a_{d-1} & a_d \\ b_0 & b_1 & \dots & b_{d-1} & * \end{array} \right\}$$

is said to be an *intersection array* of Γ. Clearly,

$$b_0 = k, b_d = c_0 = 0, c_1 = 1.$$

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = i, d(\gamma, \epsilon) = i + 1$, we see that $\Gamma_i(\gamma)$ contains k_i points, satisfying

$$k_0 = 1, k_1 = k, k_{i+1} = k_i b_i / c_{i+1} \qquad (i = 0, \dots, d-1);$$

therefore, the total number of vertices is

$$\nu = 1 + k_1 + \dots + k_d.$$

By counting edges $\{\delta, \epsilon\}$ with $d(\gamma, \delta) = 1$, we see that

$$k = a_i + b_i + c_i.$$

Examples. (i)The polygons; they have intersection array

$$\left\{\begin{matrix} * & 1\dots1 & c_d \\ 0 & 0\dots0 & 0 \\ 2 & 1\dots1 & * \end{matrix}\right\},$$

where $c_d = 2$ for the $2d$-gon and $c_d = 1$ for the $(2d+1)$-gon.
(ii)The five Platnic solids; they have intersection array

$$\left\{\begin{matrix} * & 1 \\ 0 & 2 \\ 3 & * \end{matrix}\right\}\text{(tetrahedron)}, \left\{\begin{matrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{matrix}\right\}\text{(octahedron)}, \left\{\begin{matrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{matrix}\right\}\text{(cube)},$$

$$\left\{\begin{matrix} * & 1 & 2 & 5 \\ 0 & 2 & 2 & 0 \\ 5 & 2 & 1 & * \end{matrix}\right\}\text{(icosahedron)}, \left\{\begin{matrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{matrix}\right\}\text{(dodecahedron)}.$$

3 Completely regular codes

A code C is a set of a non-empty *subset* of Γ. $C \ni x$ is said to be a codeword. The number

$$\delta(C) := \min\{d(x,y) \mid x, y \in C, x \neq y\}$$

is called the *minimum distance* of C. The distance of $x \in \Gamma$ to C is defined as

$$d(x,C) := \min\{d(x,y) \mid y \in C\},$$

and the number

$$t = t_C = \max\{d(x,C) \mid x \in \Gamma\}$$

is called the *covering radius* of C. The *subconstituents* of Γ with respected to C are the sets

$$C_l := \{x \in \Gamma \mid d(x,C) = l\};$$

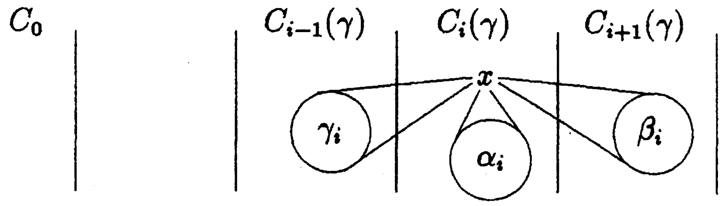
in particular,

$$C_0 = C, \qquad C_l \neq \phi \Leftrightarrow l \leq t$$

Definition 3.1 *A code C is said to be a completely regular code* (CRC) *if the numbers*

$$\gamma_i = | C_{i-1} \cap \Gamma(x) |,$$
$$\alpha_i = | C_i \cap \Gamma(x) |, \textit{and}$$
$$\beta_i = | C_{i+1} \cap \Gamma(x) |$$

are independent of the choices of $x \in C_i$.



The numbers $\gamma_i, \alpha_i, \beta_i$ are said to be an *intersection numbers* of C.

$$\iota(C) = \begin{Bmatrix} * & \gamma_1 & \dots & \gamma_{t-1} & \gamma_t \\ \alpha_0 & \alpha_1 & \dots & \alpha_{t-1} & \alpha_t \\ \beta_0 & \beta_1 & \dots & \beta_{t-1} & * \end{Bmatrix}$$

is said to be an *intersection array* of C. Clearly,

$$\beta_t = \gamma_0 = 0.$$

By counting edges $\{\delta, \epsilon\}$ with $d(\delta, C) = i$, $d(\epsilon, C) = i + 1$, we see that C_i contains κ_i points, satisfying

$$\kappa_{i+1} = \kappa_i \beta_i / \gamma_{i+1} \qquad (i = 0, \dots, t-1);$$

therefore, the total number of vertices is

$$\nu = \kappa_0 + \kappa_1 + \dots + \kappa_t.$$

By counting edges $\{\delta, \epsilon\}$ with $d(\delta, C) = i$,we see that

$$k = \alpha_i + \beta_i + \gamma_i.$$

Theorem 3.1 (Neumaier[8]) *For a completely regular code C in a connected regular graph with intersection array*

$$\iota(C) = \left\{ \begin{array}{ccccc} * & \gamma_1 & \cdots & \gamma_{t-1} & \gamma_t \\ \alpha_0 & \alpha_1 & \cdots & \alpha_{t-1} & \alpha_t \\ \beta_0 & \beta_1 & \cdots & \beta_{t-1} & * \end{array} \right\},$$

the subset C_t is a completely regular code with intersection array

$$\iota(C_t) = \left\{ \begin{array}{ccccc} * & \beta_{t-1} & \cdots & \beta_1 & \beta_0 \\ \alpha_t & \alpha_{t-1} & \cdots & \alpha_1 & \alpha_0 \\ \gamma_t & \gamma_{t-1} & \cdots & \gamma_1 & * \end{array} \right\}.$$

Theorem 3.2 (Suzuki[11,Proposition 2.1]) *In distance-regular graph, a code $\{x\}$ is a completely regular code.*

Proof. By definition of distance-regular graph, it is clear. □

We say C_t is a *reversal* of C. So, we consider the case where $2 \leq |C| \leq \frac{\nu}{2}$. We say that a completely regular code C with covering radius t in a distance regular graph Γ is trivial if $|C| \leq 1$, $|C_t| \leq 1$ or all the vertices of Γ are in C.

Theorem 3.3 (Koolen[6,Theorem15]) *Each completely regular codes C in Γ has $\gamma_i \leq \gamma_{i+1}$ and $\beta_i \geq \beta_{i+1}$, where Γ is a member of one of the following families.*

(i) *The Hamming graphs, H(n,d),*
(ii) *the Johnson graphs, J(n,d),*
(iii) *the Grassmann graphs, $G_q(n,d)$,*
(iv) *the symplectic dual polar graphs on $[C_d(q)]$,*
(v) *the orthogonal dual polar graphs on $[B_d(q)]$,*
(vi) *the orthogonal dual polar graphs on $[D_d(q)]$,*
(vii) *the orthogonal dual polar graphs on $[^2D_{d+1}(q)]$,*
(viii) *the unitary dual polar graphs on $[^2A_{2d}(r)]$,*
(ix) *the unitary dual polar graphs on $[^2A_{2d-1}(r)]$,*
(x) *the bilinear forms graphs, $H_q(n,d)$,*
(xi) *the alternating forms graphs,*
(xii) *the Hermitean forms graphs,*
(xiii) *the symmetruc bilinear forms graphs,*
(xiv) *the quadratic forms graphs,*
(xv) *the folded Johnson graphs, $\bar{J}(2m,m)$,*
(xvi) *the folded cubes,*
(xvii) *the halved cubes,*
(xviii) *the Doob graphs, the direct products of Shrikhande graphs and 4-cliques,*
(xix) *the half dual polar graphs, $D_{m,m}(q)$,*
(xx) *the Ustimenko graphs, which are the distance 1-or-2 graphs of dual polar graphs on $[C_d(q)]$, and*
(xxi) *the Hemmeter graphs, the extended bipartite doubles of the dual polar graphs on $[C_d(q)]$.*

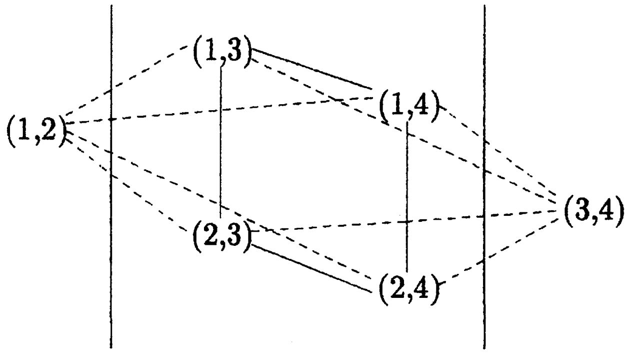
4 Completely regular codes in Johnson graph J(n,d)

Let X be a finite set of cardinality n. The Johnson graph of the d-sets in X has vertex set $\binom{n}{d}$, the collection of d-sets of X. Two vertices γ, δ are adjacent whenever $\gamma \cap \delta$ has cardinality $d-1$. The Johnson graph $J(n,d)$ has an intersection array given by

$$b_i = (n-i)(n-d-i), \quad c_i = i^2 \quad \text{for} \quad 0 \le i \le d.$$

4.1 Completely regular codes in $J(n,2)$

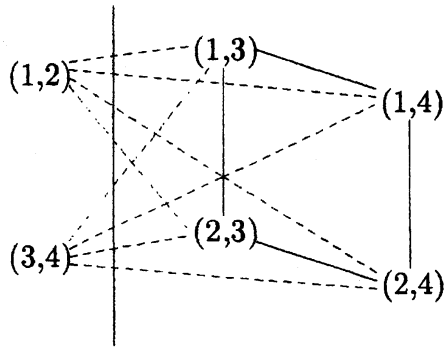
- A non-trivial CRC in Johnson graph $J(4,2)$:



$$\begin{Bmatrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{Bmatrix}$$

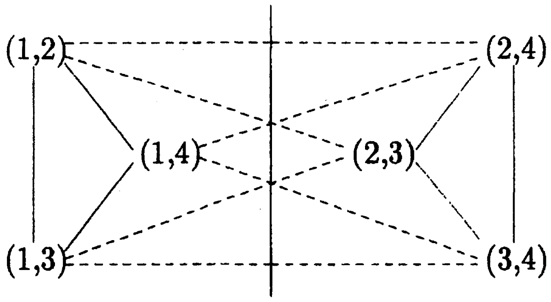
has one of the following intersection arrays.

(i) $|\, C \,| = 2$



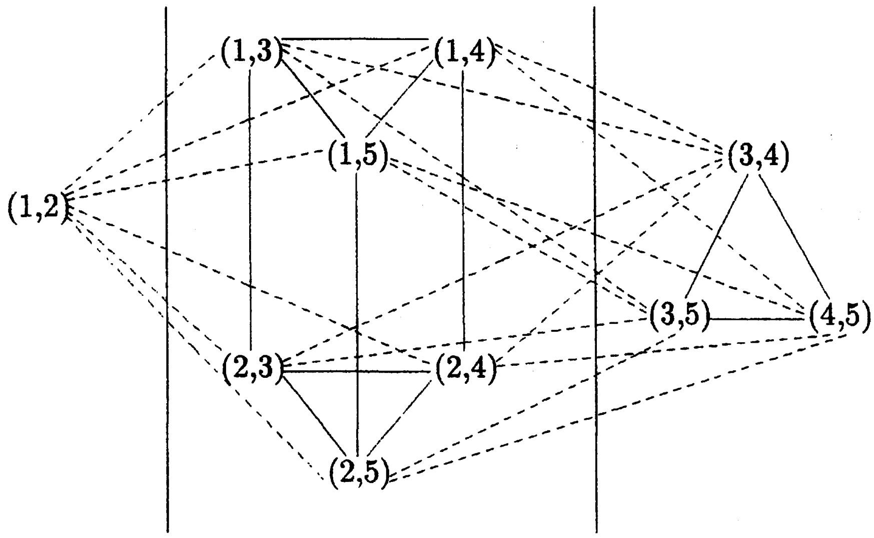
$$\begin{Bmatrix} * & 2 \\ 0 & 2 \\ 4 & * \end{Bmatrix}$$

(ii) $|C| = 3$



$$\begin{Bmatrix} * & 2 \\ 2 & 2 \\ 2 & * \end{Bmatrix}$$

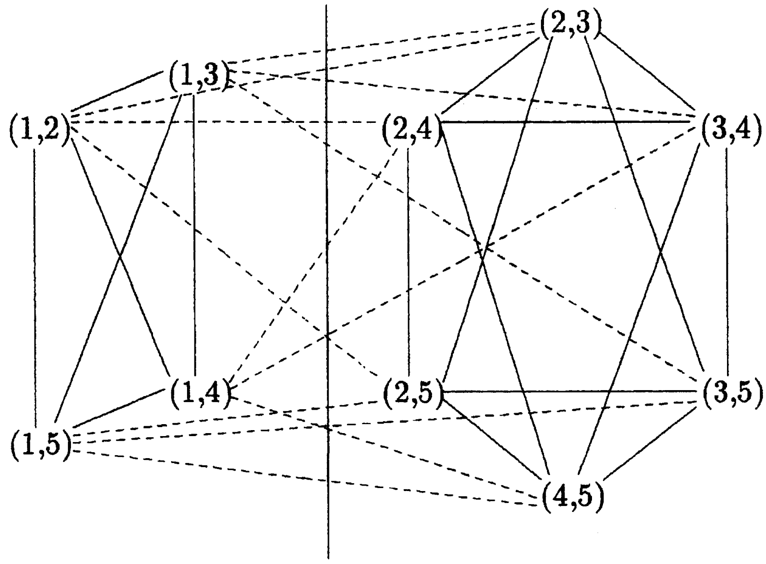
- A non-trivial CRC in Johnson graph $J(5,2)$:



$$\begin{Bmatrix} * & 1 & 4 \\ 0 & 3 & 2 \\ 6 & 2 & * \end{Bmatrix}$$

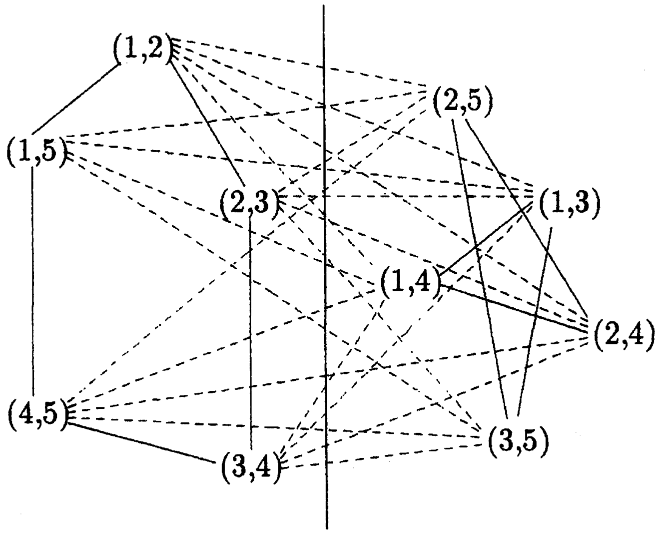
has, up to reversal, one of the following intersection arrays.

(i) $\mid C \mid = 4$



$$\begin{Bmatrix} * & 2 \\ 3 & 4 \\ 3 & * \end{Bmatrix}$$

(ii) $\mid C \mid = 5$



$$\begin{Bmatrix} * & 4 \\ 2 & 2 \\ 4 & * \end{Bmatrix}$$

- A non-trivial CRC in Johnson graph $J(6,2)$ has up to reversal, one of the following intersection arrays.

(i) $\mid C \mid = 3$ $\left\{\begin{matrix} * & 2 \\ 0 & 6 \\ 8 & * \end{matrix}\right\}, \left\{\begin{matrix} * & 2 & 6 \\ 2 & 4 & 2 \\ 6 & 2 & * \end{matrix}\right\}$

(ii) $\mid C \mid = 5$ $\left\{\begin{matrix} * & 2 \\ 4 & 6 \\ 4 & * \end{matrix}\right\}$

(iii) $\mid C \mid = 6$ $\left\{\begin{matrix} * & 4 \\ 2 & 4 \\ 6 & * \end{matrix}\right\}$

Lemma 4.1 *In Johnson graph $J(n,2)$, if n is odd, then there is no non-trivial completely regular codes with $\mid C \mid = 2$.*

Proof. For $n = 3$, there is no non-trivial completely regular codes. So, we may consider the case $n \geq 5$. If C $= \{(i,j),(i,k)\}$, we can take $(i,l),(j,m)$ in C_1, where (i,l) is adjacent (i,j) and (i,k), (j,m) is adjacent (i,j), contradiction. So, C isn't a completely regular code. If C $= \{(i,l),(j,m)\}$, we can take (i,j), (i,n) in C_1, where (i,j) is adjacent (i,l) and (j,m), (i,n) is adjacent (i,l), contradiction. So, C isn't a completely regular code. Therefore, there is no non-trivial completely regular code C with $\mid$ C $\mid = 2$. □

Lemma 4.2 *In a Johnson graph $J(n,2)$, let $C = \{(1,2),(2,3),\ldots,(n-1,n),(n,1)\}$. Then C is a completely regular code in $J(n,2)$.*

Proof. For $n = 2$ or 3, C is a trivial completely regular code. For $n = 4$, let (i,j) be in $J(n,2) \setminus$ C. Since $(i-1,i)$ or $(i,i+1)$ is in C, $d((i,j),\mathrm{C}) = 1$. Therefore (i,j) in C_1. Then, the intersention array is

$$\left\{\begin{matrix} * & 4 \\ 2 & 2(n-4) \\ 2(n-3) & * \end{matrix}\right\},$$

so C is a non-trivial completely regular code. □

4.2 Completely regular codes in $J(n, 3)$

- A non-trivial CRC in Johnson graph $J(6,3)$ has one of the following intersection arrays.

(i) $\mid C \mid = 2$ $\quad \begin{Bmatrix} * & 1 \\ 0 & 8 \\ 9 & * \end{Bmatrix}$

(ii) $\mid C \mid = 4$ $\quad \begin{Bmatrix} * & 2 \\ 1 & 7 \\ 8 & * \end{Bmatrix}, \begin{Bmatrix} * & 2 & 6 \\ 3 & 5 & 3 \\ 6 & 2 & * \end{Bmatrix}$

(iii) $\mid C \mid = 6$ $\quad \begin{Bmatrix} * & 3 \\ 2 & 6 \\ 7 & * \end{Bmatrix}$

(iv) $\mid C \mid = 8$ $\quad \begin{Bmatrix} * & 4 \\ 3 & 5 \\ 6 & * \end{Bmatrix}$

(v) $\mid C \mid = 10$ $\quad \begin{Bmatrix} * & 3 \\ 6 & 6 \\ 3 & * \end{Bmatrix}, \begin{Bmatrix} * & 5 \\ 4 & 4 \\ 5 & * \end{Bmatrix}, \begin{Bmatrix} * & 6 \\ 3 & 3 \\ 6 & * \end{Bmatrix}$

References

[1] E. Bannai and T. Ito, *Algebraic Combinatorics I*, Benjamin/Cummings, California, 1984.

[2] Biggs, N., Perfect codes in graphs, J. Combin. Th. (B) 15 (1973), 289-296.

[3] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer Verlag, Berlin, Heidelberg, 1989.

[4] Delsarte, Ph., An algebraic approach to the association scheme, Philips Res. Reports 32 (1977), 373-411.

[5] A. Jurišić, and J. Koolen and P. Terwilliger, Tight distance-regular graphs, J. Alg. Combin. 12 (2000), 163-197.

[6] J. H. Koolen, On a Conjecture of Martin on the Parameters of Completely Regular Codes and the Classification of the Completely Regular Codes in the Biggs-Smith Graph, Linear and Multilinear Algebra. 39 (1995), 3-17.

[7] W. J. Martin, Completely regular designs, J. Combin. 6 (1998), 261-273.

[8] A. Neumaier, Completely regular codes, Discrete Math, 106/107 (1992), 353-360.

[9] H. Suzuki, The Terwilliger Algebra associated with a set of vertices in a distance-regular graph, preprint.

[10] H. Suzuki, The geometric girth of a distance-regular graph having certain thin irreducible modules, preprint.

[11] H. Suzuki, On Completely Regular Codes and Related Topics, preprint.